Foundation of Cryptography (0368-4162-01), Lecture 8

Encryption Schemes

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Section 1

Definitions

Correctness

Definition 1 (encryption scheme)

A trippet of PPT's $(\mathrm{G}, \mathrm{E}, \mathrm{D})$ such that

1. $\mathrm{G}(1^n)$ outputs a key $(e, d) \in \{0,1\}^* \times \{0,1\}^*$
2. $\mathrm{E}(e, m)$ outputs a string in $c \in \{0,1\}^*$
3. $\mathrm{D}(d, c)$ outputs $m \in \{0,1\}^*$

Correctness: $\mathrm{D}(d, \mathrm{E}(e, m)) = m$, for any $(e, d) \in \mathrm{Supp}(\mathrm{G}(1^n))$ and $m \in \{0,1\}^*$

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- e – encryption key, d – decryption key
- m – plaintext, $c = \mathrm{E}(e, m)$ – ciphertext
- $E_e(m) \equiv E(e, m)$ and $D_d(c) \equiv D(d, c)$,

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- public/private key

Security

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  - Other concerns, e.g., multiple encryptions, active adversary

Semantic Security

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2. Formulate via the simulation paradigm
3. Cannot hide the message length

Semantic security – private-key model

Definition 2 (Semantic Security – private-key model)

An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ is semantically secure in the private-key model, if for any PPT A, $\exists$ PPT A' s.t. $\forall$ poly-bounded dist. ensemble $\mathcal{M} = \{\mathcal{M}_n\}_{n\in\mathbb{N}}$ and poly-bounded functions $h, f \colon \{0,1\}^* \mapsto \{0,1\}^*$

$$\Big|\Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathsf{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \\ -\Pr_{m\leftarrow\mathcal{M}_n}[\mathsf{A}'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)]\Big| = \operatorname{neg}(n)$$

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- Reflection to $\mathcal{ZK}$
- public-key variant – A gets e

Indistinguishablity of encryptions

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- Less intuitive than semantic security, but easier to work with

Indistinguishablity of encryptions – private-key model

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An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model, if for any $p, \ell \in \text{poly}$, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n \in \mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$ and poly-time B,

$$\left|\Pr_{e \leftarrow G(1^n)_1}[\mathsf{B}(z_n, E_e(x_n)) = 1] - \Pr_{e \leftarrow G(1^n)_1}[\mathsf{B}(z_n, E_e(y_n)) = 1]\right| = \text{neg}(n)$$

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- Non-uniform definition
- Public-key variant

Equivalence of definitions

Theorem 4

An encryption scheme $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ *is semantically secure iff is has indistinguishable encryptions.*

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We prove the private key case

Indistinguishability $\Longrightarrow$ Semantic Security

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Fix $\mathcal{M}$, A, f and h, be as in Definition 2.

Indistinguishability $\implies$ Semantic Security

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Algorithm 5 (A$'$)

Input: 1^n, $1^{|m|}$ and $h(m)$

1. $e \leftarrow G(1^n)_1$
2. $c = E_e(1^{|m|})$
3. Output A$(1^n, 1^{|m|}, h(m), c)$

Indistinguishability $\Longrightarrow$ Semantic Security

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Algorithm 5 (A′)

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1. $e \leftarrow G(1^n)_1$
2. $c = E_e(1^{|m|})$
3. Output $A(1^n, 1^{|m|}, h(m), c)$

Claim 6

A′ is a good simulator for A (according to Definition 2)

Proving Claim 6

For $n \in \mathbb{N}$, let

$$\delta(n) := \Big|\Pr_{m \leftarrow \mathcal{M}_n, e \leftarrow G(1^n)_1}[\mathsf{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \\ - \Pr_{m \leftarrow \mathcal{M}_n}[\mathsf{A}'(1^n, 1^{|m|}, h(1^n, m)) = f(1^n, m)]\Big|$$

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Claim 7

For every $n \in \mathbb{N}$, exists $x_n \in \mathrm{Supp}(\mathcal{M}_n)$ with

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Proof: Write the lhs and rhs terms in the definition of $\delta(n)$ as sums over the different choices of $m \in \mathrm{Supp}(\mathcal{M}_n)$, and use $|a+b| \leq |a| + |b|$

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The following algorithm contradicts the indistinguishability of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ with respect to $\{(x_n, y_n = 1^{|x_n|})\}_{n \in \mathbb{N}}$ and $\{z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n))\}_{n \in \mathbb{N}}$.

Algorithm 8 (B)

Input: $z_n = (1^n, 1^{|x_n|}, h(1^n, x_n), f(1^n, x_n)), c$
Output 1 iff $\mathsf{A}(1^n, 1^{|x_n|}, h(x_n), c) = f(1^n, x_n)$

Semantic Security $\Longrightarrow$ Indistinguishability

Assume $\exists$ PPT B, $\{x_n, y_n \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$ and a $\{z_n\}_{n\in\mathbb{N}}$, such that (wlg) for infinitely many n's:

$$\Pr_{e\leftarrow G(1^n)_1}[\mathsf{B}(z_n, E_e(x_n)) = 1] - \Pr_{e\leftarrow G(1^n)_1}[\mathsf{B}(z_n, E_e(y_n)) = 1] \geq \frac{1}{p(n)} \tag{1}$$

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- Let $\mathcal{M}_n$ be x_n wp $\frac{1}{2}$ and y_n otherwise.
- Let $f(1^n, x_n) = 1$, $f(1^n, y_n) = 0$ and $h(1^n, \cdot) = z_n$).
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$$\Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathrm{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \geq \frac{1}{2} + \frac{1}{p(n)} \tag{2}$$

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where for *any* A′

$$\Pr_{m\leftarrow\mathcal{M}_n, e\leftarrow G(1^n)_1}[\mathsf{A}(1^n, 1^{|m|}, h(1^n, m), E_e(m)) = f(1^n, m)] \leq \frac{1}{2} \tag{3}$$

Security Under Multiple Encryptions

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$\{x_{n,1}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n\in\mathbb{N}}$,
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Extensions:

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Extensions:

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- Semantic security version
- Public-key definition

Multiple Encryption in the Public-Key Model

Theorem 10

A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.

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A public-key encryption scheme has indistinguishable encryptions for multiple messages, iff it has indistinguishable encryptions for a single message.

Proof: Assume (G, E, D) is public-key secure for a single message and not for multiple messages with respect to B, $\{x_{1,t(n)}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)} \in \{0,1\}^{\ell(n)}\}_{n \in \mathbb{N}}$, $\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$.

Multiple Encryption in the Public-Key Model

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$\{z_n \in \{0,1\}^{p(n)}\}_{n \in \mathbb{N}}$.
It follows that for some function $i(n) \in [t(n)]$

$$\begin{aligned} &\big|\Pr[\mathrm{B}(1^n, e, E_e(x_{n,1}), \ldots, E_e(x_{n,i-1}), E_e(y_{n,i}) \ldots, E_e(y_{n,t(n)})) = 1] \\ -&\Pr[\mathrm{B}(1^n, e, E_e(x_{n,1}), \ldots, E_e(x_{n,i}), E_e(y_{n,i+1}) \ldots, E_e(y_{n,t(n)})) = 1]\big| \\ &> \mathrm{neg}(n) \end{aligned}$$

where in both cases $e \leftarrow \mathrm{G}(1^n)_1$

Algorithm 11 (B′)

Input: 1^n, $z_n = (i(n), x_{1,t(n)}, \ldots x_{n,t(n)}, y_{n,1}, \ldots, y_{n,t(n)}, e\ , c$
Return B($c, E_e(x_{n,1}), \ldots, E_e(x_{n,i-1}), c, E_e(y_{n,i+1}) \ldots, E_e(y_{n,t(n)})$)

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B′ is critically using the public key

Multiple Encryption in the Private-Key Model

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Assuming (non uniform) OWFs exists, there exists an encryption scheme that has private-key indistinguishable encryptions for a single messages, but not *for multiple messages*

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Construction 13

- $\mathrm{G}(1^n)$ outputs $e \leftarrow \{0,1\}^n$,
- $\mathrm{E}_e(m)$ outputs $g^{|m|}(e) \oplus m$
- $\mathrm{D}_e(c)$ outputs $g^{|c|}(e) \oplus c$

Claim 14

(G, E, D) has private-key indistinguishable encryptions for a single message

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$$\left|\Pr[\mathrm{B}(z_n, g^{|x_n|}(U_n) \oplus x_n) = 1] - \Pr[\mathrm{B}(z_n, U_{|x_n|} \oplus x_n) = 1]\right| > \mathrm{neg}(n) \tag{4}$$

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(G, E, D) does not have a private-key indistinguishable encryptions for multiple messages

Proof: Take $x_{n,1} = x_{n,2}$, $y_{n,1} \neq y_{n,2}$ and $\mathrm{D}(c_1, c_2)$ outputs 1 iff $c_1 = c_2$

Section 2

Constructions

Private key indistinguishable encryptions for multiple messages

Suffice to encrypt messages of some fixed length (here the length is n).

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- $\mathrm{G}(1^n)$: output $e \leftarrow \mathcal{F}_n$,
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Public-key indistinguishable encryptions for multiple messages

Let (G, f, Inv) be a (non-uniform) TDP, and let b be an hardcore predicate for f.

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Construction 18 (bit encryption)

- $\mathrm{G}(1^n)$: output $(e, d) \leftarrow \mathrm{G}(1^n)$
- $\mathrm{E}_e(m)$: choose $r \leftarrow \{0, 1\}^n$ and output $(y = f_e(r), c = b(r) \oplus m)$
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- We believe that public-key encryptions schemes are “more complex" than private-key ones

Section 3

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  The adversary can also ask for *decryptions* of certain messages

- In the public-key settings, the adversary is also given the public key
- We focus on indistinguishability, but each of the above definitions has an equivalent semantic security variant.

CPA Security

Let (G, E, D) be an encryption scheme. For a pair of algorithms $A = (A_1, A_2)$, $n \in \mathbb{N}$, $z \in \{0,1\}^*$ and $b \in \{0,1\}$, let:

Experiment 20 ($\mathrm{Exp}^{\mathrm{CPA}}_{A,n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow A_1^{E_e(\cdot)}(1^n, z)$
3. $c \leftarrow E_e(m_b)$
4. Output $A_2^{E_e(\cdot)}(1^n, s, c)$

CPA Security

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be an encryption scheme. For a pair of algorithms $\mathsf{A} = (\mathsf{A}_1, \mathsf{A}_2)$, $n \in \mathbb{N}$, $z \in \{0,1\}^*$ and $b \in \{0,1\}$, let:

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3. $c \leftarrow \mathsf{E}_e(m_b)$
4. Output $\mathsf{A}_2^{E_e(\cdot)}(1^n, s, c)$

Definition 21 (private key CPA)

$(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model under CPA attack, if $\forall$ PPT $\mathsf{A}_1, \mathsf{A}_2$, and poly-bounded $\{z_n\}_{n\in\mathbb{N}}$:

$$|\Pr[\mathrm{Exp}^{\mathrm{CPA}}_{\mathsf{A},n,z_n}(0) = 1] - \Pr[\mathrm{Exp}^{\mathrm{CPA}}_{\mathsf{A},n,z_n}(1) = 1]| = \mathrm{neg}(n)$$

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- The scheme from Construction 16 has indistinguishable encryptions in the private-key model under CPA attack(for short, private-key CPA secure)
- The scheme from Construction 18 has indistinguishable encryptions in the public-key model (for short, public-key CPA secure)
- In both cases, definitions are *not* equivalent

CCA Security

Experiment 22 ($\mathrm{Exp}^{\mathrm{CCA1}}_{\mathrm{A},n,z}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(m_0, m_1, s) \leftarrow \mathrm{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
3. $c \leftarrow \mathrm{E}_e(m_b)$
4. Output $\mathrm{A}_2^{E_e(\cdot)}(1^n, s, c)$

CCA Security

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3. $c \leftarrow \mathrm{E}_e(m_b)$
4. Output $\mathrm{A}_2^{E_e(\cdot)}(1^n, s, c)$

Experiment 23 ($\mathrm{Exp}^{\mathrm{CCA2}}_{\mathrm{A},n,z_n}(b)$)

1. $(e, d) \leftarrow G(1^n)$
2. $(x_0, x_1, s) \leftarrow \mathrm{A}_1^{E_e(\cdot), D_d(\cdot)}(1^n, z)$
3. $c \leftarrow \mathrm{E}_e(x_b)$
4. Output $\mathrm{A}_2^{E_e(\cdot), D_d^{\neg c}(\cdot)}(1^n, s, c)$

Definition 24 (private key CCA1/CCA2)

$(\mathsf{G}, \mathsf{E}, \mathsf{D})$ has indistinguishable encryptions in the private-key model under $x \in \{\mathrm{CCA1}, \mathrm{CCA2}\}$ attack, if $\forall$ PPT $\mathsf{A}_1, \mathsf{A}_2$, and poly-bounded $\{z_n\}_{n\in\mathbb{N}}$:

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- The public key definition is analogous

Private-key CCA2

- Is the scheme from Construction 16 private-key CCA1 secure?

Private-key CCA2

- Is the scheme from Construction 16 private-key CCA1 secure?
- CCA2 secure?

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a private key CPA scheme, and let $(\mathsf{Gen}_M, \mathsf{Mac}, \mathsf{Vrfy})$ be an existential unforgeable strong MAC.

Construction 25

- $\mathsf{G}'(1^n)$: Output $(e \leftarrow \mathsf{G}_E(1^n), k \leftarrow \mathsf{Gen}_M(1^n))$.[a]
- $\mathsf{E}'_{d,k}(m)$: let $c = \mathsf{E}_e(m)$ and output $(c, t = \mathsf{Mac}_k(c))$
- $\mathsf{D}_{e,k}(c, t)$: if $\mathsf{Vrfy}_k(c, t) = 1$, output $\mathsf{D}_e(c)$. Otherwise, output $\perp$

[a] We assume for simplicity that the encryption and decryption keys are the same.

Private-key CCA2

- Is the scheme from Construction 16 private-key CCA1 secure?
- CCA2 secure?

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Theorem 26

Construction 25 is a private-key CCA2-secure encryption scheme.

Theorem 26

Construction 25 is a private-key CCA2*-secure encryption scheme.*

Proof: ?

Public-key CCA1

Public-key CCA1

Let $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ be a public-key CPA scheme and let (P, V) be a NIZK for $\mathcal{L} = \{(c_0, c_1, pk_0, pk_1) \colon \exists (m, z_0, z_1) \;\; s.t. \; c_0 = \mathsf{E}_{pk_0}(m, z_0) \wedge c_1 = \mathsf{E}_{pk_1}(m, z_1)\}$

Public-key CCA1

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Construction 27 (The Naor-Yung Paradigm)

- $\mathrm{G}'(1^n)$:
  1. For $i \in \{0,1\}$: set $(sk_i, pk_i) \leftarrow \mathrm{G}(1^n)$.
  2. Let $r \leftarrow \{0,1\}^{\ell(n)}$, and output $pk' = (pk_0, pk_1, r)$ and $sk' = (pk', sk_0, sk_1)$
- $\mathrm{E}'_{pk'}(m)$:
  1. For $i \in \{0,1\}$: $c_i = \mathrm{E}_{pk_i}(m, z_i)$, where z_i is a uniformly chosen string of the right length
  2. $\pi \leftarrow \mathrm{P}((c_0, c_1, pk_0, pk_1), (m, z_0, z_1), r)$
  3. Output (c_0, c_1, π).
- $\mathrm{D}'_{sk'}(c_0, c_1, \pi)$: If $\mathrm{V}((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, return $\mathrm{D}_{sk_0}(c_0)$. Otherwise, return $\perp$

Omitted details:

- We assume for simplicity that the encryption key output by G(1^n) is of length at least n.
- ℓ is an arbitrary polynomial, and determines the maximum message length to encrypt using ”security parameter" n.

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Is the scheme CCA1 secure?

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Is the scheme CCA1 secure? We need the NIZK to be “adaptive secure".

Theorem 28

Assuming that (P, V) *is adaptive secure, then Construction 27 is a public-key* $\mathsf{CCA1}$ *secure encryption scheme.*

Omitted details:

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Is the scheme CCA1 secure? We need the NIZK to be “adaptive secure".

Theorem 28

Assuming that (P, V) *is adaptive secure, then Construction 27 is a public-key* CCA1 *secure encryption scheme.*

Proof: Given an attacker A' for the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$, we use it to construct an attacker A on the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$.
Let $\mathsf{S} = (\mathsf{S}_1, \mathsf{S}_2)$ be the (adaptive) simulator for $(\mathsf{P}, \mathsf{V}, \mathcal{L})$

Algorithm 29 (A)

Input: $(1^n, pk)$

1. let $j \leftarrow \{0,1\}$, $pk_{1-j} = pk$, $(pk_j, sk_j) \leftarrow \mathrm{G}(1^n)$ and $(r,s) \leftarrow \mathrm{S}_1(1^n)$
2. Emulate $\mathrm{A}'(1^n, pk' = (pk_0, pk_1, r))$ as follows:
3. On query (c_0, c_1, π) of A' to D':
   If $\mathrm{V}((c_0, c_1, pk_0, pk_1), \pi, r) = 1$, answer $\mathrm{D}_{sk_j}(c_j)$.
   Otherwise, answer $\perp$.
4. Output the same pair (m_0, m_1) as A' does
5. On challenge c $(= \mathrm{E}_{pk}(m_b))$:
   - Set $c_{1-j} = c$, $a \leftarrow \{0,1\}$, $c_j = \mathrm{E}_{pk_j}(m_a)$, and $\pi \leftarrow \mathrm{S}_2((c_0, c_1, pk_0, pk_1), r, s)$
   - Send $c' = (c_0, c_1, \pi)$ to A'
6. Output the same value that A' does

Claim 30

Assume that A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ with probability $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ with probability $(\delta(n) - \mathsf{neg}(n))/2$.

Claim 30

Assume that A' breaks the CCA1 security of $(\mathsf{G}', \mathsf{E}', \mathsf{D}')$ with probability $\delta(n)$, then A breaks the CPA security of $(\mathsf{G}, \mathsf{E}, \mathsf{D})$ with probability $(\delta(n) - \operatorname{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[\mathsf{A}' \text{ “makes" } \mathsf{A}(1^n) \text{ decrypt an invalid cipher}] = \operatorname{neg}(n) \quad (5)$$

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Hence, only negligible information leaks about j.

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Hence, only negligible information leaks about j.
Let $\mathrm{A}'(1^n, a^*, b^*)$ be the output of $\mathrm{A}'(1^n)$ in the emulation induced by A, where $a = a^*$ and $b = b^*$.

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The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

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Hence, only negligible information leaks about j.
Let $A'(1^n, a^*, b^*)$ be the output of $A'(1^n)$ in the emulation induced by A, where $a = a^*$ and $b = b^*$. It holds that

1. $A'(1^n, 0, 1) \equiv A'(1^n, 1, 0)$

Claim 30

Assume that A' breaks the CCA1 security of (G', E', D') with probability $\delta(n)$, then A breaks the CPA security of (G, E, D) with probability $(\delta(n) - \mathrm{neg}(n))/2$.

The adaptive soundness and adaptive zero-knowledge of (P, V), yields that

$$\Pr[A' \text{ “makes" } A(1^n) \text{ decrypt an invalid cipher}] = \mathrm{neg}(n) \qquad (5)$$

Hence, only negligible information leaks about j.
Let $A'(1^n, a^*, b^*)$ be the output of $A'(1^n)$ in the emulation induced by A, where $a = a^*$ and $b = b^*$. It holds that

1. $A'(1^n, 0, 1) \equiv A'(1^n, 1, 0)$
2. The adaptive zero-knowledge of (P, V) yields that $|\Pr[A'(1^n, 1, 1) = 1] - \Pr[A'(1^n, 0, 0) = 1]| \geq \delta(n) - \mathrm{neg}(n)$

Let A(b) be the outputs of A when the challenge is b.

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$$\begin{aligned}
&|\Pr[A(1) = 1] - \Pr[A(0) = 1]| \\
&\quad = |\frac{1}{2}(\Pr[A'(0,1) = 1] + \Pr[A'(1,1) = 1]) \\
&\qquad - \frac{1}{2}(\Pr[A'(0,0) = 1] + \Pr[A'(1,0) = 1])|
\end{aligned}$$

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&\quad \geq \frac{1}{2}|\Pr[A'(1,1) = 1] - \Pr[A'(0,0) = 1]| \\
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&\quad \geq \frac{1}{2}|\Pr[A'(1,1) = 1] - \Pr[A'(0,0) = 1]| \\
&\qquad - \frac{1}{2}|\Pr[A'(1,0) = 1] - \Pr[A'(0,1) = 1]| \\
&\quad \geq (\delta(n) - \mathrm{neg}(n))/2
\end{aligned}
$$

Public-key CCA2

- Is Construction 27 CCA2 secure?

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- **Problem:** Soundness might not hold with respect to the simulated CRS, after seeing a proof for an *invalid* statement

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- **Solution:** use *simulation sound* NIZK