Application of Information Theory, Lecture 9

Parallel Repetition of Interactive Arguments

Handout Mode

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December 15, 2011

Part I

Interactive Proofs and Arguments

$\mathcal{NP}$ as a Non-interactive Proofs

Definition 1 ($\mathcal{NP}$)

$\mathcal{L} \in \mathcal{NP}$ iff $\exists$ and poly-time algorithm V such that:

- $\forall x \in \mathcal{L}$ there exists $w \in \{0,1\}^*$ s.t. $\mathsf{V}(x, w) = 1$
- $\mathsf{V}(x, w) = 0$ for every $x \notin \mathcal{L}$ and $w \in \{0,1\}^*$

Only $|x|$ counts for the running time of V.

This proof system has

- Efficient verifier, efficient prover (given the witness)
- Soundness holds unconditionally

Interactive proofs/arguments

Protocols between efficient verifier and unbounded/efficent prover.

Definition 2 (Interactive proof)

A protocol (P, V) is an interactive proof for $\mathcal{L}$, if V is a PPT and:

Completeness $\forall x \in \mathcal{L}$: $\Pr[(\mathsf{P}, \mathsf{V})(x) = 1] \geq 2/3$.

Soundness $\forall x \notin \mathcal{L}$, and any algorithm P^*: $\Pr[(\mathsf{P}^*, \mathsf{V})(x) = 1] \leq 1/3$.

IP is the class of languages that have interactive proofs.

- $\mathsf{IP} = \mathrm{PSPACE}$!
- The above protocol has completeness error $\frac{1}{3}$, and sourness error $\frac{1}{3}$
- We typically consider achieve (directly) perfect completeness.
- Smaller “soundness error" achieved via repetition.
- Relaxation: interactive arguments [also known as, Computationally sound proofs]: soundness only guaranteed against efficient (PPT) provers.
- Games — no-input protocols.

Section 1

Interactive Proof for Graph Non-Isomorphism

Graph isomorphism

Π_m – the set of all permutations from $[m]$ to $[m]$

Definition 3 (graph isomorphism)

Graphs $G_0 = ([m], E_0)$ and $G_1 = ([m], E_1)$ are isomorphic, denoted $G_0 \equiv G_1$, if $\exists \pi \in \Pi_m$ such that
$(u, v) \in E_0$ iff $(\pi(u), \pi(v)) \in E_1$.

- $\mathcal{GI} = \{(G_0, G_1) \colon G_0 \equiv G_1\} \in \mathcal{NP}$
- Does $\mathcal{GNI} = \{(G_0, G_1) \colon G_0 \not\equiv G_1\} \in \mathcal{NP}$?
- We will show a simple interactive proof for $\mathcal{GNI}$
  Idea: Beer tasting...

Interactive proof for $\mathcal{GNI}$

Protocol 4 $((\mathsf{P},\mathsf{V})(\mathsf{G}_0 = ([m], E_0), \mathsf{G}_1 = ([m], E_1)))$

1. V chooses $b \leftarrow \{0,1\}$ and $\pi \leftarrow \Pi_m$, and sends $\pi(E_b)$ to P.[a]
2. P send b' to V (tries to set $b' = b$).
3. V accepts iff $b' = b$.

[a] $\pi(E) = \{(\pi(u), \pi(v) \colon (u,v) \in E\}$.

Claim 5

The above protocol is IP for $\mathcal{GNI}$, with perfect completeness and soundness error $\frac{1}{2}$.

Proving Claim 5

- Graph isomorphism is an equivalence relation (separates all graph pairs into separate subsets)
- $([m], \pi(E_i))$ is a random element in $[\mathrm{G}_i]$ — the equivalence class of G_i

Hence,

$\mathrm{G}_0 \equiv \mathrm{G}_1$: $\Pr[b' = b] \leq \frac{1}{2}$.

$\mathrm{G}_0 \not\equiv \mathrm{G}_1$: $\Pr[b' = b] = 1$ (i.e., P can, possibly inefficiently, extracted from $\pi(E_i)$)

□

Part II

Hardness Amplification

Hardness amplification

- In most settings we need very small soundness error (i.e., close to 0)
- Typically done by "amplifying the security" of an interactive proof/argument of large soundness error.
- Two main approaches:
  - Sequential repetition: achieves optimal amplification rate in almost any computation model, but increases the round complexity
  - Parallel repetition: sometimes does not achieve optimal amplification rate and sometimes achieves nothing
- How come parallel repetition might not work? Example
- Parallel repetition does achieve optimal amplification rate for interactive proofs and public-coin interactive arguments
- Public-coin interactive proof/argument — in each round the verifier flips coins and sends them to the prover. To compute its output, the verifier applies some (fixed) function to the protocol's transcript.

Hardness amplification, cont.

- Give a protocol $\pi = (\mathsf{P}, \mathsf{V})$ and $k \in \mathbb{N}$, let $\pi^{(k)} = (\mathsf{P}^{(k)}, \mathsf{V}^{(k)})$ be the k-fold parallel repetition of π: i.e., k parallel independent copies of π
- Assume $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \leq \varepsilon$ for any s-size algorithm $\widetilde{\mathsf{P}}$, we would like to prove that $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] \leq f^{(k)}(\varepsilon)$ for any $s^{(k)}$-size algorithm $\widetilde{\mathsf{P}^{(k)}}$.
- Typically, $s^{(k)} = s \cdot \mathrm{poly}(f^{(k)}(\varepsilon)/k)$
- If $f(\varepsilon) = \varepsilon^{\Omega(k)}$, the above is an exponential-rate amplification (and hence optimal)
- If $f(\varepsilon) = \varepsilon^{\delta_1 \cdot k^{\delta_2}}$, the above is a weakly-exponential-rate amplification
- Why size?
- Concrete security
- In the following we focus on games (no input protocols)

Section 2

Parallel repetition of public-coin interactive argument

Parallel repetition of public-coin interactive argument

Theorem 6

Let $\pi = (\mathsf{P}, \mathsf{V})$ be m-round, public-coin protocol with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \leq \varepsilon$ for any s-size $\widetilde{\mathsf{P}}$, then $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] \leq \varepsilon^{k/4}$ for any $s \cdot \frac{\varepsilon^{k/4}}{mk^3 s_{\mathsf{V}}}$-size $\widetilde{\mathsf{P}^{(k)}}$, where s_{V} is V's size.

Proof plan: Let $\widetilde{\mathsf{P}^{(k)}}$ be $s^{(k)}$-size algorithm with $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}) = 1^k\right] = \varepsilon^{(k)}$, we construct $s^{(k)} \cdot \frac{mk^3 s_{\mathsf{V}}}{\varepsilon^{(k)}}$-size $\widetilde{\mathsf{P}}$ with $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq (\varepsilon^{(k)})^{4/k}$.

- The $k/4$ in the exponent can be pushed to be almost k.
- Assume for simplicity that $\widetilde{\mathsf{P}^{(k)}}$ is deterministic
- Assume wlg. that V sends the first message in π and that in each round it sends ℓ coins.
- We view the coins of $\mathsf{V}^{(k)}$ as a matrix $R \in \{0,1\}^{m \times (k\ell)}$, letting R_j denote the coins of the j'th round
- Let $x^j = x_1, \ldots, x_j$ (hence R^j denote the coins used in the first j rounds).
- Let $\mathbf{R} \sim \{0,1\}^{m \times (k\ell)}$

Algorithm $\widetilde{\mathsf{P}}$

Let $q = k^2$.

Algorithm 7 ($\widetilde{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. Upon getting the j'th round message r from V, do:

   2.1 Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned that $R_{1,\ldots,j-1} = \widetilde{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = r$.

   2.2 If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$:

   2.2.1 Set $\widetilde{R}_j = R_j$

   2.2.2 Send a_{j,i^*} back to V, for a_j being the j'th message $\widetilde{\mathsf{P}^{(k)}}$ send to $\mathsf{V}^{(k)}$ in $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R))$.

   Else, GOTO Line 2.1

   2.3 Abort, if overall number of sampling exceeds $\lceil qm/\varepsilon^{(k)} \rceil$.

- Let $\widetilde{\mathsf{P}}'$ be the non aborting variant of $\widetilde{\mathsf{P}}$, let $\widetilde{\mathbf{R}}$ and $\widetilde{\mathbf{N}}$ be the value of $\widetilde{R}$ and $\#$ of samples done in a random execution of $(\widetilde{\mathsf{P}}', \mathsf{V}^{(k)})$, respectively.
- $\Pr\left[(\widetilde{\mathsf{P}}, \mathsf{V}) = 1\right] \geq \Pr\left[\mathrm{win}(\widetilde{\mathbf{R}}, \widetilde{\mathbf{N}}) := (\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widetilde{\mathbf{R}})) = 1^k \wedge \widetilde{\mathbf{N}} \leq qm/\varepsilon^{(k)}\right]$.

Ideal "attacker"

Experiment 8 ($\hat{\mathsf{P}}$)

For $j = 1$ to m:

1. Let $R \leftarrow \{0,1\}^{m\times(k\ell)}$, conditioned that $R_{1,\ldots,j-1} = \hat{R}_{1,\ldots,j-1}$.
2. If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\hat{R}_j = R_j$. Else, GOTO Line 1.

- Let $\hat{\mathbf{R}}$ be the value of $\hat{R}$ in the end of a random execution of $\hat{\mathsf{P}}$.
- $\hat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R}))=1^k}$
- In particular, $\Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\hat{\mathbf{R}}) = 1^k\right] = 1$
- Let $\hat{\mathbf{N}}$ be $\#$ of samples done in $\hat{\mathsf{P}}$.

Lemma 9

$\Pr\left[\hat{\boldsymbol{N}} > qm/\varepsilon^{(k)}\right] < \frac{1}{q}$

Hence, $\Pr\left[\mathrm{win}(\widehat{\mathbf{R}}, \widehat{\mathbf{N}})\right] = \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widehat{\mathbf{R}})) = 1^k \wedge \widehat{\mathbf{N}} \leq qm/\varepsilon^{(k)}\right] \geq 1 - \frac{1}{q}$

Proving Lemma 9 — $\Pr\left[\hat{\mathbf{N}} > qm/\varepsilon^{(k)}\right] < \frac{1}{q}$

- Let $(X_1, \ldots, X_m) = \mathbf{R}$ and $(Y_1, \ldots, Y_m) = \widehat{\mathbf{R}}$
- For $\mathbf{y} \in \mathrm{Supp}(Y^j)$, let
  $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathrm{P}^{(k)}}, \mathrm{V}^{(k)}(X^m) = 1^k \mid X^j = \mathbf{y}\right]$
- Conditioned on $Y^j = \mathbf{y}$, the expected $\#$ of samples done in $(j+1)$'th round of $\widehat{\mathrm{P}}$ is $\frac{1}{v(\mathbf{y})}$.
- We prove Lemma 9 showing that $\mathrm{E}\left[\frac{1}{v(Y^j)}\right] \leq \frac{1}{\varepsilon^{(k)}}$ for every $j \in \{0, \ldots, m-1\}$

Claim 10

For $j \in \{0, \ldots, m-1\}$ and $\mathbf{y} \in \mathrm{Supp}(Y^j)$, it holds that $\Pr_{Y^j}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Hence, $\mathrm{E}\left[\frac{1}{v(Y^j)}\right] = \sum_{\mathbf{y} \in \mathrm{Supp}(Y^j)} \Pr[Y^j = \mathbf{y}] \cdot \frac{1}{v(\mathbf{y})}$
$= \sum_{\mathbf{y}} \Pr[X^j = \mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}} \cdot \frac{1}{v(\mathbf{y})} = \frac{1}{\varepsilon^{(k)}} \cdot \sum_{\mathbf{y} \in \mathrm{Supp}(Y^j)} \Pr[X^j = \mathbf{y}] \leq \frac{1}{\varepsilon^{(k)}}$. $\square$

Proving **Claim 10** — $\Pr_{Y^j}[\mathbf{y}] = \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}$

Recall $v(\mathbf{y}) := \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(X^m) = 1^k \mid X^j = \mathbf{y}\right]$. Note that

$$\Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] = \sum_{\ell=1}^{\infty}(1 - v(\mathbf{y}_{1,\dots,j-1}))^{\ell-1} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] \cdot v(\mathbf{y}) \qquad (1)$$

$$= \frac{1}{v(\mathbf{y}_{1,\dots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] \cdot v(\mathbf{y})$$

The proof proceeds by induction on j.

$$\begin{aligned}
\Pr_{Y^j}[\mathbf{y}] &= \Pr_{Y^{j-1}}\left[\mathbf{y}_{1,\dots,j-1}\right] \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] \\
&= \Pr_{X^{j-1}}\left[\mathbf{y}_{1,\dots,j-1}\right] \cdot \frac{v(\mathbf{y}_{1,\dots,j-1})}{\varepsilon^{(k)}} \cdot \Pr_{Y_j|Y^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] && \text{(i.h.)} \\
&= \Pr_{X^{j-1}}\left[\mathbf{y}_{1,\dots,j-1}\right] \cdot \frac{v(\mathbf{y}_{1,\dots,j-1})}{\varepsilon^{(k)}} \cdot \frac{v(\mathbf{y})}{v(\mathbf{y}_{1,\dots,j-1})} \cdot \Pr_{X_j|X^{j-1}=\mathbf{y}_{1,\dots,j-1}}[y_j] && \text{(Eq. (1))} \\
&= \Pr_{X^j}[\mathbf{y}] \cdot \frac{v(\mathbf{y})}{\varepsilon^{(k)}}.
\end{aligned}$$

From ideal to real

Let $\widetilde{\mathsf{I}}$ be the value of i^* in $\widetilde{\mathsf{P}}$.

Claim 11

$D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i})$.

Claim 12

$\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathsf{I}}=i}) \leq D(\widehat{\mathbf{R}}||\mathbf{R})$.

1. Thm. 7 in Lecture 7 $\implies D(\widehat{\mathbf{R}}||\mathbf{R}) \leq \log \frac{1}{\Pr[(\widetilde{\mathsf{P}^{(k)},\mathsf{V}^{(k)}})(\mathbf{R}))=1^k]} \leq \log\frac{1}{\varepsilon^{(k)}}$
2. Hence, $D(\mathrm{win}(\widehat{\mathbf{R}},\widehat{\mathbf{N}})||\,\mathrm{win}(\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})) \leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq -\frac{1}{k}\cdot\log\varepsilon^{(k)}$
3. Lemma 15 $\implies \alpha := \Pr[\mathrm{win}(\widehat{\mathbf{R}},\widehat{\mathbf{N}})] \geq 1-\frac{1}{q}$, and let $\beta := \Pr[\mathrm{win}(\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})]$.
4. By (2), $\alpha\cdot\log\frac{\alpha}{\beta} + (1-\alpha)\log\frac{1-\alpha}{1-\beta} \leq -\frac{1}{k}\cdot\log\varepsilon^{(k)}$

   $\implies \beta \geq 2^{\log\alpha+\frac{1-\alpha}{\alpha}\log(1-\alpha)+\frac{1}{\alpha k}\log\varepsilon^{(k)}}$
5. Since $q=k^2$: $\alpha \geq 2^{-\frac{2}{q}} \geq 2^{-\frac{1}{k}}$ and $\frac{1-\alpha}{\alpha}\log(1-\alpha) \geq -\frac{4\log k}{k^2} \geq -\frac{1}{k}$
6. We conclude that $\beta \geq 2^{\frac{4}{k}\log\varepsilon^{(k)}} = \sqrt[k/4]{\varepsilon^{(k)}}$.$\square$

Proving Claim 12 — $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i}) \leq D(\widehat{\mathbf{R}}||\mathbf{R})$

Lemma 13

Let $Z = \{Z_{ij}\}_{(i,j)\in[k]\times[m]}$ be iids and let W be an event. For $z \in \mathrm{Supp}(Z)$, let
$\xi_i(z) :=$
$\prod_{j=1}^{m} \Pr[Z_{j,i} = z_{i,j}] \cdot \Pr[Z_{j,-i} = z_{i,j-1} | Z_{1,\ldots,j-1} = z_{1,\ldots,j-1} \wedge Z_{j,i} = z_{i,j} \wedge W]$.
Then $\sum_{i=1}^{k} D(Z|_W||\xi_i) \leq D(Z|_W||Z)$.

Letting $Z = \mathbf{R}$ and W be the event $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k$, Lemma 13 yields that $\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}_{\widetilde{\mathbf{I}}=i}) = \sum_{i\in[k]} D(\mathbf{R}|_W||\widetilde{\mathbf{R}}_{\widetilde{\mathbf{I}}=i}) \leq D(\mathbf{R}|_W||\mathbf{R}) = D(\widehat{\mathbf{R}}||\mathbf{R})$. $\square$

Proving Lemma 13

We prove for $m = k = 2$.

$Z = (X_0, X_1, Y_0, Y_1)$ iids and W an event.

$\xi_i(x_0, x_1, y_0, y_1) := \Pr[X_i = x_i] \cdot \Pr[X_{\bar{i}} = x_{\bar{i}} \mid X_i = x_i \wedge W] \cdot \Pr[Y_i = y_i] \cdot \Pr[Y_{\bar{i}} = Y_{\bar{i}} \mid Y_i = y_i \wedge (X_0, X_1) = (x_0, x_1) \wedge W]$.

We need to prove that $\sum_{i=1}^{2} D(Z|_W || \xi_i) \leq D(Z|_W || Z)$.

- Let $U = p_Z$ and $C = p_{Z|_W}$.
- Let $X = (X_0, X_1)$
- $Q(x_0, x_1, y_0, y_1) := \Pr[X_0 = x_0 | W] \cdot \Pr[X_1 = x_1 | W] \cdot \Pr[Y_0 = y_0 | W, X = (x_0, x_1)] \cdot \Pr[Y_1 = y_1 | W, X = (x_0, x_1)]$
- We write $\frac{C(x_0,x_1,y_0,y_1)}{U(x_0,x_1,y_0,y_1)} = \frac{\Pr[X_0=x_0|W]\cdot\Pr[Y_0=y_0|W,X=(x_0,x_1)]}{\Pr[X_0=x_0]\cdot\Pr[Y_0=y_0]} \cdot \frac{\Pr[X_1=x_1|W]\cdot\Pr[Y_1=y_1|W,X=(x_0,x_1)]}{\Pr[X_1=x_1]\cdot\Pr[Y_1=y_1]} \cdot \frac{C(x_0,x_1,y_0,y_1)}{Q(x_0,x_1,y_0,y_1)}$

Proving Lemma 13, cont.

$$D(C||U) = \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_0 = x_0|W]\cdot\Pr[Y_0 = y_0|W, X = (x_0,x_1)]}{\Pr[X_0 = x_0]\cdot\Pr[Y_0 = y_0]}\right]$$
$$+ \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{\Pr[X_1 = x_1|W]\cdot\Pr[Y_1 = y_1|W, X = (x_0,x_1)]}{\Pr[X_1 = x_1]\cdot\Pr[Y_1 = y_1]}\right]$$
$$+ \mathop{\mathrm{E}}_{(x_0,x_1,y_0,y_1)\leftarrow C}\left[\log \frac{C(x_0,x_1,y_0,y_1)}{Q(x_0,x_1,y_0,y_1)}\right].$$

It follows that

$$\begin{aligned}
D(C||U) &= D(X_0|_W, X_1|_{W,X_0}, Y_0|_{W,X}, Y_1|_{W,X,Y_0}||X_0, X_1|_{W,X_0}, Y_0, Y_1|_{W,X,Y_0}) \\
&\quad + D(X_1|_W, X_1|_{W,X_1}, Y_1|_{W,X}, Y_1|_{W,X,Y_1}||X_1, X_1|_{W,X_1}, Y_1, Y_1|_{W,X,Y_1}) \\
&\quad + D(C||Q) \\
&= \sum_{i=1}^{2} D(Z|_W||\xi_i) + D(C||Q) \\
&\geq \sum_{i=1}^{2} D(Z|_W||\xi_i). \square
\end{aligned}$$

Ideal "attacker", variant

Experiment 14 ($\widehat{\mathsf{P}}$)

1. Let $i^* \leftarrow [k]$.
2. For $j = 1$ to m:
   - **2.1** Let $R \leftarrow \{0,1\}^{m \times (k\ell)}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$.
   - **2.2** If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_{j,i^*} = R_{j,i^*}$. Else, GOTO Line 2.1.
   - **2.3** Let $R \leftarrow \{0,1\}^{m \times \ell}$, conditioned on $R_{1,\ldots,j-1} = \widehat{R}_{1,\ldots,j-1}$ and $R_{j,i^*} = \widehat{R}_{j,i^*}$.
   - **2.4** If $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(R)) = 1^k$, set $\widehat{R}_j = R_j$. Else, GOTO Line 2.3.

- Let $\widehat{\mathbf{R}}$ be the final value of $\widehat{R}$ in $\widehat{\mathsf{P}}$.
- $\widehat{\mathbf{R}} \sim \mathbf{R}|_{(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\mathbf{R})) = 1^k}$
- Let $\widehat{\mathbf{N}}$ be the $\#$ of Step-2.3-samples done in $\widehat{\mathsf{P}}$.

Lemma 15 (essentially the same proof as of Lemma 9)

$$\Pr\left[\mathrm{win}(\widehat{\boldsymbol{R}}, \widehat{\boldsymbol{N}})\right] = \Pr\left[(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)}(\widehat{\boldsymbol{R}})) = 1^k \wedge \widehat{\boldsymbol{N}} \leq qm/\varepsilon^{(k)}\right] \geq 1 - \frac{1}{q}$$

Proving Claim 11 — $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}||\widetilde{\mathbf{R}}|_{\widetilde{\mathbf{I}}=i})$

Let $\widehat{\mathbf{I}}$ be the value of i^* in $\widehat{\mathrm{P}}$ (recall that $\widetilde{\mathbf{I}}$ is the value of i^* in $\widetilde{\mathrm{P}}$).

Let $(\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) = (\widetilde{\mathbf{R}},\widetilde{\mathbf{N}})|_{\widetilde{\mathbf{I}}=i}$ and $(\widehat{R}_{(i)},\widehat{\mathbf{N}}_{(i)}) = (\widehat{\mathbf{R}},\widehat{\mathbf{N}})|_{\widehat{\mathbf{I}}=i}$. Note that $\widehat{R}_{(i)} = \widehat{\mathbf{R}}$.

$$
\begin{aligned}
D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) &\leq D(\widehat{\mathbf{R}},\widehat{\mathbf{N}},\widehat{\mathbf{I}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}},\widetilde{\mathbf{I}}) && \text{(data-processing)}\\
&= D(\widehat{\mathbf{I}}||\widetilde{\mathbf{I}}) + \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) && \text{(chain rule)}\\
&= \frac{1}{k}\sum_{i\in[k]} D(\widehat{\mathbf{R}}_{i},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)})
\end{aligned}
$$

For $i \in [k]$, it holds that

$$
\begin{aligned}
D(\widehat{\mathbf{R}}_{(i)},\widehat{\mathbf{N}}_{(i)}||\widetilde{\mathbf{R}}_{(i)},\widetilde{\mathbf{N}}_{(i)}) &= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) + \mathop{\mathrm{E}}_{r\leftarrow\widehat{\mathbf{R}}_{(i)}}\left[D(\widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r}||\widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r})\right] && \text{(chain ru}\\
&= D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)}) \quad \text{(since } \widehat{\mathbf{N}}_{(i)}|_{\widehat{\mathbf{R}}_{(i)}=r} \equiv \widetilde{\mathbf{N}}_{(i)}|_{\widetilde{\mathbf{R}}_{(i)}=r}\text{)}
\end{aligned}
$$

Hence, $D(\widehat{\mathbf{R}},\widehat{\mathbf{N}}||\widetilde{\mathbf{R}},\widetilde{\mathbf{N}}) \leq \frac{1}{k}\sum_{\in[k]} D(\widehat{\mathbf{R}}_{(i)}||\widetilde{\mathbf{R}}_{(i)})$ □

Parallel repetition of interactive proofs

- Similar proof to the public-coin proof we gave above.
- In each round, the attacker $\widetilde{\mathsf{P}}$ samples random continuations of $(\widetilde{\mathsf{P}^{(k)}}, \mathsf{V}^{(k)})$, till he gets an accepting execution.
- Why fails us to extend this approach for non-public-coin interactive arguments?

Section 3

Parallel amplification for any interactive argument

Parallel amplification theorem for any protocol

- Can we amplify the security of any interactive argument "in parallel"?
- Yes we can!

Relevant papers

- Kai-Min Chung and Rafael Pass: Tight Parallel Repetition Theorems for Public-Coin Arguments using KL-divergence.

  The proof given in class is in the spirit of this paper.